Trajectories with constant acceleration, jerk, etc. in 2D Minkowski spacetime

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Abstract: In this work we study one spatial dimension motion kinematics in Minkowski spacetime. First, we discuss different approaches to the accelerated observer. Then, we propose a parametrization for the four-velocity and solve exactly the case that describes the asymptotic behaviour for constant higher order derivatives of velocity. Finally, we examine how more general cases change our solution.

I. INTRODUCTION

One spatial dimension motion is the simplest case to be considered in kinematics and it is the basis for extensions to higher spatial dimensions. The study of constant acceleration, jerk, etc. trajectories in Galilean mechanics amounts to the integration of a trivial differential equation. Yet, this is not true in the Einsteinian case.
Uniformly accelerated motion is basic in some well-known physical processes as Unruh effect and, consequently, it has been highly studied but it is often presented under a single perspective. In the first place, this work aims to exhibit some different views on how to derive the result. Still, kinematics is not finished yet. Higher-order position derivatives are commonly ignored. Those kinematical objects are the main subjects of study of this work. First, Frenet-Serret frame and the correct definitions for the kinematical objects are presented, which lead to the equations for uniform acceleration, jerk, etc. We get analytic solutions for the case that describes the asymptotic behaviour; and numerical for some more general cases.
Since our interest is in one spatial dimension motion, the work is all done in 1+1-dimensional Minkowski spacetime with metric $\eta = diag(-1, 1)$ and it is set $c = 1$.

II. CONSTANT ACCELERATION

A. The standard derivation

Let's suppose that $x(t)$ describes the trajectory of an observer with constant proper acceleration and p is a point in that trajectory corresponding to coordinate time $\tilde{t}_0$. The observer is at rest with himself. Therefore, the acceleration at point p will be measured as the derivative of the velocity with respect to the time coordinate of the inertial reference system in which he is at rest. Then, we consider the boost that makes the observer be at rest at p. Hence, its velocity parameter must counterbalance the velocity of the observer at p given in the original system, $\dot{x}(\tilde{t}_0)$ (where dot denotes derivative with respect t). The boost is defined by,

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma(t - vx) \end{cases} \tag{1}$$

where $v = \dot{x}(\tilde{t}_0)$ and $\gamma = \frac{1}{\sqrt{1-v^2}}$. What we need to compute is $\frac{d^2x'}{dt'^2}|_p$. To do so, we use the chain rule $\frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt} = \frac{1}{\dot{t}'}\frac{d}{dt}$. Taking into account that $\dot{x}'|_p = 0$,

$$\begin{aligned} \frac{d^2x'}{dt'^2}\Big|_p &= \frac{d}{dt'}(\frac{dx'}{dt'})\Big|_p = \frac{1}{\dot{t}'}\frac{d}{dt}(\frac{1}{\dot{t}'}\frac{dx'}{dt})\Big|_p = \\ &= \frac{1}{\dot{t}'}\frac{d}{dt}(\frac{\dot{x}'}{\dot{t}'})\Big|_p = \frac{1}{\dot{t}'}\frac{\ddot{x}'\dot{t}' - \dot{x}'\ddot{t}'}{(\dot{t}')^2}\Big|_p = \frac{\ddot{x}'}{(\dot{t}')^2}\Big|_p \end{aligned} \tag{2}$$

From (1) we obtain $\ddot{x}' = \gamma\ddot{x}$ and $\dot{t}'|_p = \gamma(1 - v^2) = \frac{1}{\gamma}$, hence,

$$\frac{d^2x'}{dt'^2}\Big|_p = \ddot{x}\gamma^3\Big|_p = \frac{\ddot{x}(\tilde{t}_0)}{[1 - \dot{x}^2(\tilde{t}_0)]^{\frac{3}{2}}} \tag{3}$$

Taking into account that we are studying constant acceleration, the differential equation at any time will be

$$a = \frac{\ddot{x}(t)}{[1 - \dot{x}^2(t)]^{\frac{3}{2}}} \tag{4}$$

The general solution to (4) is

$$x = x_0 \pm \sqrt{(t - t_0)^2 + \frac{1}{a^2}} \tag{5}$$

Finally, this expression can be written as the standard hyperbola equation

$$(x - x_0)^2 - (t - t_0)^2 = \frac{1}{a^2} \tag{6}$$

B. Pure relativistic formalism

Four-position is defined by $\boldsymbol{X} \equiv \begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix}$. Therefore, we can calculate four-velocity and four-acceleration as its first and second derivatives with respect to proper time τ, respectively [1]. To do so, we use $\gamma = \frac{dt}{d\tau}$.

$$\boldsymbol{U} \equiv \frac{d\boldsymbol{X}}{d\tau} = \frac{d\boldsymbol{X}}{dt}\frac{dt}{d\tau} = \begin{pmatrix} \gamma \\ \gamma\dot{x} \end{pmatrix} \tag{7}$$

$$\boldsymbol{A} \equiv \frac{d^2\boldsymbol{X}}{d\tau^2} = \frac{d\boldsymbol{U}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{U}}{dt} = \begin{pmatrix} \gamma\dot{\gamma} \\ \gamma\dot{\gamma}\dot{x} + \gamma^2\ddot{x} \end{pmatrix} \tag{8}$$

For an observer at rest we have $\boldsymbol{A} = \begin{pmatrix} 0 \\ a \end{pmatrix}$. Therefore, $A_\mu A^\mu = a^2$. Since $A_\mu A^\mu$ is the same in all reference frames,

$$A_\mu A^\mu = -\gamma^2\dot{\gamma}^2 + (\gamma\dot{\gamma}\dot{x} + \gamma^2\ddot{x})^2 = \frac{\ddot{x}^2}{(1-\dot{x}^2)^3} = a^2 \quad (9)$$

Hence, we recover (4) with solutions (5) and (6).

C. The 'Differential boosts' method

We can picture acceleration as the sum of tiny boosts with a velocity parameter δv. Since $\delta x = x' - x$ and $\delta t = t' - t$, using (1) and considering $(\delta v)^2 \to 0$, we find the relations $\delta x = t\delta v$ and $\delta t = x\delta v$. These relations lead to,

$$\begin{cases} \frac{dx}{dv} = t \\ \frac{dt}{dv} = x \end{cases} \quad (10)$$

The general solution to (10) is

$$\begin{cases} t = C_1 e^v + C_2 e^{-v} \\ x = C_1 e^v - C_2 e^{-v} \end{cases} \quad (11)$$

where C_1 and C_2 are integration constants. Setting $t(v = 0) = 0$ we get $C_1 = -C_2$. In order to find the physical meaning of C_1 we can make the Galilean approximation (where, with our initial conditions, we know that $x = x_0 + \frac{1}{2}at^2$).

$$x = \sqrt{4C_1^2 + t^2} = 2C_1\sqrt{1 + \frac{t^2}{4C_1^2}} \simeq 2C_1(1 + \frac{1}{2}\frac{t^2}{4C_1^2}) = \\ = 2C_1 + \frac{1}{2}\frac{t^2}{2C_1} \quad (12)$$

Therefore, from comparison we find $C_1 = \frac{1}{2a}$. Finally, the parametric solution is

$$\begin{cases} x = \frac{1}{a}\cosh(v) \\ t = \frac{1}{a}\sinh(v) \end{cases} \quad (13)$$

As we can always perform a translation, the solution is (6).

D. Acceleration's boost invariance

Acceleration is the rate of change of velocity. This implies that it must be the same for every boost we apply. Therefore, if a trajectory is boost invariant, it must be a uniformly accelerated observer. But boost invariant trajectories are easy to build if one considers that the origin of coordinates is boost invariant and Minkowski lenght is also boost invariant. Since the distance between a point and the origin is boost invariant,

$$x^2 - t^2 = x'^2 - t'^2 = C \quad (14)$$

the trajectory of this uniformly accelerated observer will be an hyperbola. To find the value of C, we can proceed with the Galilean approximation (as done before). Doing so, we find $C = \frac{1}{a^2}$. Once again, applying a translation of coordinates, we recover (6).

III. BEYOND CONSTANT ACCELERATION

A. Frenet-Serret frame in 1+1-Minkowski space

In order to define a Frenet-Serret frame in Minkowski space it is convenient to use the same construction used in Euclidean space [2]. To include the Minkowski metric in the construction it must be imposed $\mathbf{v}_1^2 = -1$ and $\mathbf{v}_2^2 = 1$ (the generalitzation to arbitrary dimensions is made by $\mathbf{v}_i^2 = 1$, $\forall i > 1$). Since $0 = \frac{1}{2}\frac{d}{d\tau}(\mathbf{v}_1 \cdot \mathbf{v}_1) = \frac{d\mathbf{v}_1}{d\tau} \cdot \mathbf{v}_1$, $\frac{d\mathbf{v}_1}{d\tau}$ is perpendicular to $\mathbf{v}_1$. Therefore, we define the first curvature invariant, κ, as the modulus of $\frac{d\mathbf{v}_1}{d\tau}$ and $\mathbf{v}_2$ as the unit vector defined by its direction,

$$\mathbf{v}_2 = \frac{1}{\kappa}\frac{d\mathbf{v}_1}{d\tau} \quad (15)$$

Now, taking the derivatives of $\mathbf{v}_2 \cdot \mathbf{v}_2 = 1$ and $\mathbf{v}_1 \cdot \mathbf{v}_2$, we find the relations $\frac{d\mathbf{v}_2}{d\tau} \cdot \mathbf{v}_2 = 0$ and $\frac{d\mathbf{v}_2}{d\tau} \cdot \mathbf{v}_1 = -\mathbf{v}_2 \cdot \frac{d\mathbf{v}_1}{d\tau}$. Using them, we can prove $\frac{d\mathbf{v}_2}{d\tau} - \kappa\mathbf{v}_1$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\begin{cases} (\frac{d\mathbf{v}_2}{d\tau} - \kappa\mathbf{v}_1)\mathbf{v}_2 = \frac{d\mathbf{v}_2}{d\tau}\mathbf{v}_2 - \kappa\mathbf{v}_1\mathbf{v}_2 = 0 \\ (\frac{d\mathbf{v}_2}{d\tau} - \kappa\mathbf{v}_1)\mathbf{v}_1 = -\kappa\mathbf{v}_2\mathbf{v}_2 + \kappa = 0 \end{cases} \quad (16)$$

Its modulus would define κ_2. However, because we are working in 1+1 dimensions, that vector has to be $\vec{0}$. Hence,

$$\frac{d\mathbf{v}_2}{d\tau} = \kappa\mathbf{v}_1 \quad (17)$$

Therefore, Frenet-Serret equations in 1+1-Minkowski spacetime are (15) and (17).

B. Position's higher order derivatives

When an observer at rest starts moving all velocity's derivatives are different from zero. Nevertheless, they are usually omitted. Here we aim to study these cases. In classical mechanics, acceleration's derivative is called jerk, $\boldsymbol{\Sigma}$, and jerk's derivative is called snap, $\boldsymbol{\Xi}$.
Let $\boldsymbol{P}_n$ be the kinematical term related to velocity's nth derivative, thus $\boldsymbol{P}_1 \equiv \boldsymbol{A}$, $\boldsymbol{P}_2 \equiv \boldsymbol{\Sigma}$, etc. We adopt the $\boldsymbol{P}_n$ definition of Russo and Townsend [3],

$$\boldsymbol{P}_{n+1} \equiv \left(\frac{d\boldsymbol{P}_n}{d\tau}\right)_\perp = \frac{d\boldsymbol{P}_n}{d\tau} - (\boldsymbol{A}\cdot\boldsymbol{P}_n)\boldsymbol{U} \quad (18)$$

This definition satisfies that $\boldsymbol{P}_n$ is space-like for $n \geq 1$ and that $\boldsymbol{P}_n = 0$ implies $\boldsymbol{P}_{n+1} = 0$. Identifying $\mathbf{v}_1 = \boldsymbol{U}$ and $\mathbf{v}_2 = \hat{\boldsymbol{A}}$ in (15) and (17), where $\hat{\boldsymbol{A}}$ is the unit vector in the direction of acceleration (using prime as τ's derivative),

$$\begin{cases} \boldsymbol{U}' = \kappa \hat{\boldsymbol{A}} \\ \hat{\boldsymbol{A}}' = \kappa \boldsymbol{U} \end{cases} \tag{19}$$

Hence, $\boldsymbol{P}_1 = \boldsymbol{A} = \kappa\hat{\boldsymbol{A}}$ and $\kappa = |\boldsymbol{A}|$. We can easily show by induction the general term (we use ${}^{(n)}$ to denote τ's nth derivative),

$$\boldsymbol{P}_n = \kappa^{(n-1)}\hat{\boldsymbol{A}} \tag{20}$$

To prove it, we first calculate $\boldsymbol{P}_2$,

$$\begin{aligned} \boldsymbol{P}_2 = \frac{d\boldsymbol{A}}{d\tau} - \boldsymbol{A}^2\boldsymbol{U} = \frac{d(\kappa\hat{\boldsymbol{A}})}{d\tau} - (\kappa\hat{\boldsymbol{A}})^2\boldsymbol{U} = \\ \kappa'\hat{\boldsymbol{A}} + \kappa^2\boldsymbol{U} - \kappa^2\boldsymbol{U} = \kappa'\hat{\boldsymbol{A}} \end{aligned} \tag{21}$$

Then, assuming (20) we compute $\boldsymbol{P}_{n+1}$,

$$\begin{aligned} \boldsymbol{P}_{n+1} = \frac{d\boldsymbol{P}_n}{d\tau} - (\boldsymbol{A}\boldsymbol{P}_n)\boldsymbol{U} = \frac{d(\kappa^{(n-1)}\hat{\boldsymbol{A}})}{d\tau} - (\kappa\kappa^{(n-1)})\boldsymbol{U} = \\ = \kappa^{(n)}\hat{\boldsymbol{A}} + \kappa^{(n-1)}\kappa\boldsymbol{U} - \kappa\kappa^{(n-1)}\boldsymbol{U} = \kappa^{(n)}\hat{\boldsymbol{A}} \end{aligned} \tag{22}$$

what was to be demonstrated.
The following parametrization turns out to be very useful to get the contents of (20),

$$\boldsymbol{U} = \begin{pmatrix} t'(\tau) \\ x'(\tau) \end{pmatrix} = \begin{pmatrix} \cosh(f(\tau)) \\ \sinh(f(\tau)) \end{pmatrix} \tag{23}$$

This parametrization satisfies two important properties. Since $t'(\tau) = \cosh(f(\tau))$ is always positive, coordinate time will only go forward. Furthermore, the norm is conserved, $\boldsymbol{U}^2 = \sinh^2(f(\tau)) - \cosh^2(f(\tau)) = -1$. Computing the acceleration, applying (19), we can find $\kappa = |f'(\tau)|$,

$$\boldsymbol{P}_1 = \frac{d\boldsymbol{U}}{d\tau} = f'(\tau)\hat{\boldsymbol{A}} = \pm\kappa\hat{\boldsymbol{A}} \tag{24}$$

Hence, using (20),

$$\boldsymbol{P}_n = f^{(n)}(\tau)\hat{\boldsymbol{A}} \tag{25}$$

We obtain the remarkable result that if $\boldsymbol{P}_n$ is constant, $f(\tau)$ will be a polynomial of degree n and $\boldsymbol{P}_{n+1}$ vanishes.

C. Exact solutions for monomials

We want to study kinematics with constant $\boldsymbol{P}_n$. The asymptotic behavior is well described by the higher power in $f(\tau)$. Hence, we reduce it to a monomial,

$$f(\tau) = \left(\frac{\tau}{\tau_0}\right)^n \tag{26}$$

Therefore, constant $\boldsymbol{P}_n$ motion can be solved by integrating (23) with (26),

$$\boldsymbol{U} = \begin{pmatrix} t'(\tau) \\ x'(\tau) \end{pmatrix} = \begin{pmatrix} \cosh\left[\left(\frac{\tau}{\tau_0}\right)^n\right] \\ \sinh\left[\left(\frac{\tau}{\tau_0}\right)^n\right] \end{pmatrix} \tag{27}$$

The solution to (27) is,

$$\begin{cases} t(\tau) = t_0 + \tau\, {}_1F_2\left(\frac{1}{2n}; \frac{1}{2}, \frac{2n+1}{2n}; \frac{1}{4}\left(\frac{\tau}{\tau_0}\right)^{2n}\right) \\ x(\tau) = x_0 + \frac{\tau^{n+1}}{(n+1)\tau_0^n}\, {}_1F_2\left(\frac{n+1}{2n}; \frac{3}{2}, \frac{3n+1}{2n}; \frac{1}{4}\left(\frac{\tau}{\tau_0}\right)^{2n}\right) \end{cases} \tag{28}$$

Where $x_0 = x(0)$ and $t_0 = t(0)$.
This expression can be simplified in some particular cases. For constant acceleration ($\boldsymbol{P}_2 = 0$) is recovered the same solution found previously,

$$\begin{cases} t(\tau) = t_0 + \tau_0 \sinh(\frac{\tau}{\tau_0}) = t_0 + \frac{1}{a}\sinh(a\tau) \\ x(\tau) = x_0 + \tau_0(\cosh(\frac{\tau}{\tau_0}) - 1) = x_0 + \frac{1}{a}(\cosh(a\tau) - 1) \end{cases} \tag{29}$$

For constant jerk ($\boldsymbol{P}_3 = 0$) the solution can be written in terms of error functions,

$$\begin{cases} t(\tau) = t_0 + \tau_0\frac{\sqrt{\pi}}{4}(\text{erfi}(\frac{\tau}{\tau_0}) + \text{erf}(\frac{\tau}{\tau_0})) \\ x(\tau) = x_0 + \tau_0\frac{\sqrt{\pi}}{4}(\text{erfi}(\frac{\tau}{\tau_0}) - \text{erf}(\frac{\tau}{\tau_0})) \end{cases} \tag{30}$$

Solving the equations numerically we find Fig.(1).

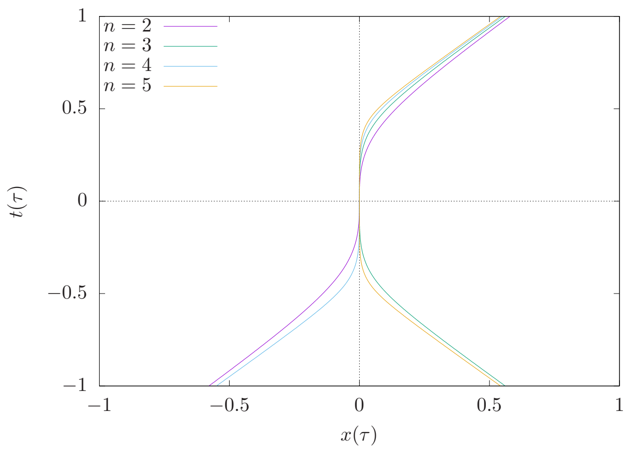

FIG. 1: Numerical solutions to different values of n. Initial conditions are set to $x_0 = t_0 = 0$. For $\tau \to \pm\infty$, trajectories are tangent to the light cone.

The fact that even and odd values of n tend to $-\infty$ or ∞ for $\tau \to -\infty$, respectively, is related to the fact that physical velocity of the observer with respect to coordinate time is $v(\tau) = \frac{dx(\tau)}{dt(\tau)} = \tanh[(\frac{\tau}{\tau_0})^n] \simeq (\frac{\tau}{\tau_0})^n$ (for small $(\frac{\tau}{\tau_0})^n$). When n is even, $v \geq 0$. Therefore, as in Fig.(1) $x_0 = t_0 = 0$, $x(\tau < 0) < 0$. However, when n is odd, $v(\tau < 0) < 0$, which implies $x(\tau < 0) > 0$.

It is possible to find the asymptotes (and horizons) analytically. Let us suppose $x_0 = t_0 = 0$, defining

$$\begin{cases} I_n^{(1)}(u) \equiv \int_0^u \exp(v^n) dv \\ I_n^{(2)}(u) \equiv \int_0^u \exp(-v^n) dv \end{cases} \tag{31}$$

the solutions to (27) can be written as,

$$\begin{cases} t(\tau) = \frac{\tau_0}{2}\left[I_n^{(1)}(\frac{\tau}{\tau_0}) + I_n^{(2)}(\frac{\tau}{\tau_0})\right] \\ x(\tau) = \frac{\tau_0}{2}\left[I_n^{(1)}(\frac{\tau}{\tau_0}) - I_n^{(2)}(\frac{\tau}{\tau_0})\right] \end{cases} \tag{32}$$

These functions satisfy the identities: $I_n^{(2)}(\infty) = \Gamma\left(\frac{n+1}{n}\right)$ for all $n \in \mathbb{N}$; $I_n^{(2)}(-\infty) = -\Gamma\left(\frac{n+1}{n}\right)$ for even n and $I_n^{(1)}(-\infty) = -\Gamma\left(\frac{n+1}{n}\right)$ for odd n. The proof is provided in the appendix.
For $\tau > 0$, the asymptote must be $t = x + C$. Hence, applying (32) and making the limit $\tau \to \infty$, it follows,

$$C = \lim_{u\to\infty} \tau_0^2\, I_n^{(2)}(u) = \tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{33}$$

$$t = x + \tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{34}$$

For $\tau < 0$, we have to distinguish between even and odd n.
If n is even, the asymptote has the form $t = x + C'$. Therefore,

$$C' = \lim_{u\to-\infty} \tau_0^2\, I_n^{(2)}(u) = -\tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{35}$$

$$t = x - \tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{36}$$

If n is odd, the asymptote is $t = -x + C''$. Therefore,

$$C'' = \lim_{u\to-\infty} \tau_0^2\, I_n^{(1)}(u) = -\tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{37}$$

$$t = -x - \tau_0^2\, \Gamma\left(\frac{n+1}{n}\right) \tag{38}$$

The τ_0^2 comes from the change of variables $u = \frac{\tau}{\tau_0}$ in $I_n^{(1)}(u)$ and $I_n^{(2)}(u)$.
From the physical point of view, for n even, those asymptotes imply that if we simultaneously emit a photon at $\tau \to -\infty$ from the position of the observer, at $\tau \to \infty$ the time ellapsed between the arrival of the photon and the observer will be $\Delta t = 2\tau_0^2\Gamma(\frac{n+1}{n})$, which is finite.

D. Discussion on more general cases

For $\tau \approx 0$ the monomial is no longer dominant and the full polynomial must be considered. Despite not being possible to solve the equations analytically, we can study some general properties by analyzing $f(\tau)$. In particular, as commented previously, $v(\tau) = \tanh(f(\tau))$. Therefore, finding the zeros for $f(\tau)$ will provide us the zeros for $v(\tau)$. Consequently, we will have information about certain turning points. If the polynomial has a region with positive and negative values, physical velocity will also be positive or negative. In Fig.(2) there are some numerical examples.

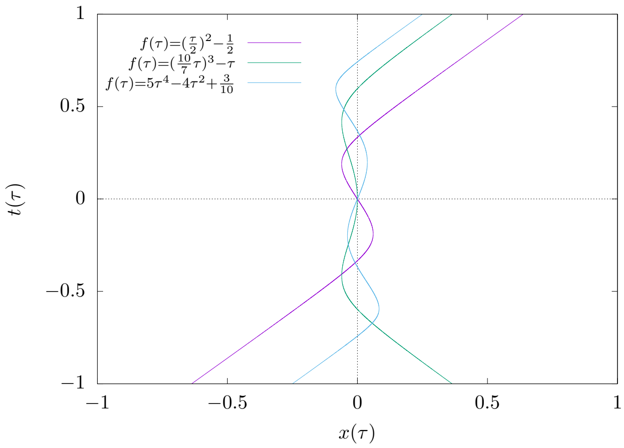

FIG. 2: Numerical solutions to the trajectories associated to some polynomials.

E. Boost on the monomials solutions

In terms of rapidity, ϕ, a boost is written as,

$$\Lambda(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \tag{39}$$

Hence, using hyperbolic trigonometry identities, we find that four-velocity in the new reference frame, U', will be,

$$U' = \Lambda(\phi)U = \begin{pmatrix} \cosh(f(\tau) - \phi) \\ \sinh(f(\tau) - \phi) \end{pmatrix} \tag{40}$$

Therefore, a boost changes the constant term of the polynomial.

In the monomial case, it is possible to find the zeros in order to predict the behaviour near the origin,

$$\tau = \phi^{\frac{1}{n}}\tau_0 \tag{41}$$

For $n = 2$, there will be two zeros if $\phi > 0$ and none if $\phi < 0$, as can be seen in Fig.(3).
Furthermore, in the monomial case, asymptotes can be computed in terms of rapidity. We previously found that the asymptotes on the monomial case with no boost are,

$$\begin{cases} t = x + \tau_0^2\,\Gamma\left(\frac{n+1}{n}\right), & \forall n \\ t = x - \tau_0^2\,\Gamma\left(\frac{n+1}{n}\right), & n \text{ even} \\ t = -x - \tau_0^2\,\Gamma\left(\frac{n+1}{n}\right), & n \text{ odd} \end{cases} \tag{42}$$

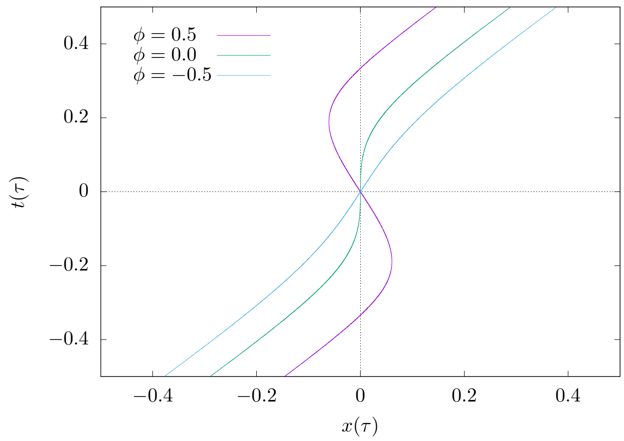

FIG. 3: Numerical solution to different boosts when $n = 2$.

In order to express the asymptotes in a boosted frame we can use the Lorentz transormation in terms of rapidity, (39). Therefore, the asymptotes in the boosted frame are,

$$\begin{cases} t' = x' + \tau_0^2\, e^{\phi}\,\Gamma\left(\frac{n+1}{n}\right), & \forall n \\ t' = x' - \tau_0^2\, e^{\phi}\,\Gamma\left(\frac{n+1}{n}\right), & n \text{ even} \\ t' = -x' - \tau_0^2\, e^{\phi}\,\Gamma\left(\frac{n+1}{n}\right), & n \text{ odd} \end{cases} \tag{43}$$

IV. CONCLUSIONS

- The kinematics of uniformly accelerated observer in Minkowski space can be deduced in several ways. In particular we study the standard change of reference method, the pure relativistic formalism, acceleration as a sum of infinitesimal boosts and the invariance of acceleration under boosts. Nevertheless, this does not exclude the existence of other approaches. This presentation may also serve as a pedagogical introduction to this subject.
- In order to study velocity's higher order derivatives, we introduce a parametrization for four-velocity, (23), which satisfies the basic requirements needed. Then, using Frenet-Serret frame and a proper definition of $\boldsymbol{P}_n$, (18), we find its relation with our function, $f(\tau)$. Having a constant $\boldsymbol{P}_n$ motion implies $f(\tau)$ is a polynomial and $\boldsymbol{P}_{n+1}$ vanishes.
- We study the monomial case for constant $\boldsymbol{P}_n$ motion and found the analytical solution, (28), and its asymptotes. Then, we introduce how more general cases would modify our result and we prove that a boost just modifies the constant term of the polynomial of $f(\tau)$. We also obtain the asymptotes for the monomial case with a boost.

V. APPENDIX

In order to avoid excessive irrelevant calculations, the integrals on (33), (35) and (37) are computed here.
To prove $\lim_{x\to\infty} I_n^{(2)}(x) = \Gamma\left(\frac{n+1}{n}\right)$, we start integrating by parts $\Gamma\left(\frac{n+1}{n}\right)$,

$$\begin{aligned} \Gamma\left(\frac{n+1}{n}\right) &\equiv \int_0^\infty v^{\frac{1}{n}} e^{-v} dv = \\ &= \left[-v^{\frac{1}{n}} e^{-v}\right]_0^\infty - \int_0^\infty \frac{1}{n} v^{\frac{1}{n}-1}(-e^{-v}) dv = \\ &= \int_0^\infty \frac{1}{n} v^{\frac{1}{n}-1} e^{-v} dv \end{aligned} \tag{44}$$

By the change of variables $v = u^n$, $dv = nu^{n-1}du$, the integral becomes,

$$\int_0^\infty \frac{1}{n}(u^n)^{\frac{1-n}{n}} e^{-u^n} nu^{n-1} du = \int_0^\infty e^{-u^n} du = \lim_{x\to\infty} I_n^{(2)}(x) \tag{45}$$

After this result, we can prove the two results on (35) and (37) by making the change of variables $w = -u$, $dw = -du$ on $\int_0^\infty e^{-u^n} du$;

$$\Gamma\left(\frac{n+1}{n}\right) = -\int_0^{-\infty} e^{-(-w)^n} dw \tag{46}$$

Since $(-1)^n$ is 1 for even n and -1 for odd n, we can identify,

$$\begin{cases} \lim\limits_{x\to-\infty} I_n^{(1)}(x) = -\Gamma\left(\frac{n+1}{n}\right), & n \text{ odd} \\ \lim\limits_{x\to-\infty} I_n^{(2)}(x) = -\Gamma\left(\frac{n+1}{n}\right), & n \text{ even} \end{cases} \tag{47}$$

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